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On viscous Burgers-like equations with linearly growing initial data

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1. Introduction

We consider a viscous Burgers-like equation of the form

$$\text{(E)} \quad \begin{cases} \partial_t u - \Delta u + \operatorname{div}\mathbf{G}(u) = 0 & \text{in } \mathbf{R}^n \times (0,T), \\ u|_{t=0} = u_0 & \text{in } \mathbf{R}^n, \end{cases}$$

where $\partial_t = \partial/\partial t$. It is well-known that if u_0 is bounded, (E) admits a unique global solution (cf. [8]). In this paper we consider the case that u_0 is not bounded at the space infinity. This paper specifies the growth of nonlinear term as $\mathbf{G}(r) \sim r^2$ for large r. A typical example is the viscous Burgers equation. Our goal is to solve the initial value problem when the initial data may grow linearly at the space infinity. We shall prove that the problem admits a unique local regular solution. The global existence is not expected in general even for $n = 1$ since $u(x,t) = -x/(1-t)$ is a solution of the viscous Burgers equation: $\partial_t u - \Delta u + u\partial_x u = 0$ with $u_0(x) = -x$, where $\partial_x = \partial/\partial x$. We also obtain an optimal estimate of the existence time. In fact, the existence time interval $(0,T)$ is estimated from below by a constant multiple over a Lipschtz bound for initial data, $T \geq C_2\|\nabla u_0\|_\infty$; here the constant C_2 is estimated by the structure of $\mathbf{G}$, and $\|\nabla u_0\|_\infty$ is defined by $\|\nabla u_0\|_\infty = \left(\sum_{i=1}^n \|\partial_i u_0\|_\infty^2\right)^{1/2}$, where $\partial_i u_0 = \partial u_0/\partial x_i$.

To state our main result precisely we assume the following bounds for $\mathbf{G} = (G_1, \cdots, G_n) \in C^{2+\alpha}(\mathbf{R};\mathbf{R}^n)$ with some $\alpha \in (0,1)$:

$$\text{(C)} \qquad \begin{aligned} C_1 &:= \sup_i \sup_{r\in\mathbf{R}} \frac{|G_i'(r)|}{\langle r\rangle} < \infty, \\ C_2 &:= \left(\sum_{i=1}^n \left(\sup_{r\in\mathbf{R}} |G_i''(r)|\right)^2\right)^{1/2} < \infty, \\ C_3 &:= \sup_i \sup_{r_1,r_2\in\mathbf{R}} \frac{|G_i''(r_1) - G_i''(r_2)|}{|r_1 - r_2|^\alpha} < \infty. \end{aligned}$$

Here we set $\langle x\rangle = \sqrt{1+|x|^2}$ for $x \in \mathbf{R}^n$ and G_i' is denotes the derivative of G_i. A typical example satisfying this assumption (C) is $G_i(r) = r^2$ $(1 \leq i \leq n)$. We prepare a few function spaces allowing growth at space infinity. Let L_m^p be of the form

$$L_m^p = L_m^p(\mathbf{R}^n) = \left\{ f \in L_{loc}^p(\mathbf{R}^n) \;\middle|\; \|f\|_{p,m} := \left\| \frac{f(x)}{\langle x\rangle^m} \right\|_p < \infty) \right\}.$$

Of course, $L^p_0 = L^p$ by definition so that $\|\cdot\|_{p,0} = \|\cdot\|_p$. Let X_B be of the form

$$X_B = \left\{ f \in C^1(\mathbf{R}^n) \ \mid \ \|f\|_{X_B} := \|f\|_{\infty,1} + \|\nabla f\|_\infty < \infty \right\}.$$

Definition. *By a classical solution u of* (E) *we mean that $u \in C(\mathbf{R}^n \times [0,T))$ is C^2 in space and C^1 in time, and it solves* (E).

Theorem *(Existence and uniqueness of a solution of a viscous Burgers like equation). Assume that $\mathbf{G} \in C^{2+\alpha}(\mathbf{R};\mathbf{R}^n)$ satisfies bounds (C). Assume that $u_0 \in X_B$. Then there exist $T \geq T_0 := \dfrac{1}{C_2\|\nabla u_0\|_\infty}$ and $u \in L^\infty_{loc}([0,T); L^\infty_1(\mathbf{R}^n)) \cap C(\mathbf{R}^n \times [0,T))$ that satisfies (E) in $\mathbf{R}^n \times (0,T)$ with $u|_{t=0} = u_0$. The existence time estimate $T \geq T_0$ is optimal in the sense that a classical solution may not exist in $[0,T)$ for $T > T_0$.*

Optimality is easily observed by the next example.

Example. We set $\xi, \eta \in \mathbf{R}^n$ and we take

$$\mathbf{G}(r) = \frac{\xi}{2} r^2,$$

so that (E) becomes

$$(\mathrm{E})' \qquad \begin{cases} \partial_t u - \Delta u + (\nabla u \bullet \xi) u = 0, \\ u|_{t=0} = u_0, \end{cases}$$

where "$\bullet$" is the inner product. Then the function

$$u(x,t) = \frac{(\xi \bullet \eta)(\eta \bullet x)}{1 + (\xi \bullet \eta)t}$$

solves $(\mathrm{E})'$ with the initial condition $u_0(x) = \eta \bullet x$. If $\xi \bullet \eta < 0$, the solution of $(\mathrm{E})'$ blows up at $t = 1/|\xi \bullet \eta|$. Since $C_2 = |\xi|$, $\|\nabla u_0\|_\infty = |\eta|$, this example shows the estimate $T \leq T_0$ is optimal if ξ parallels η.

Remark. (1) It is easy to see that this existence time estimate is invariant under a rotation of space variables x. If we do not care about rotation invariance of results, there is a sharper estimate for T by defining T_0 by

$$T_0 = \left(\inf_{\frac{1}{p}+\frac{1}{q}=1} \left(\sum_{i=1}^n |C_2^{(i)}|^p \right)^{1/p} \left(\sum_{i=1}^n \|\partial_i u_0\|_\infty^q \right)^{1/q} \right)^{-1},$$

where $C_2^{(i)} = \sup_{r\in\mathbf{R}} |G_i''(r)|$, $1 \le p \le \infty$, $1 \le q \le \infty$.

(2) If we consider $\partial_t u - \varepsilon\Delta u + \mathrm{div}\mathbf{G}(u) = 0$ for $\varepsilon > 0$ instead of the evolution equation (E), we still obtain the existence time estimate $T \ge T_0$ independent of $\varepsilon > 0$. This is easily follows from our theorem by changing the variable t by s/ε or x by $y/\sqrt{\varepsilon}$.

For the viscous Burgers equation:

$$\text{(B)} \qquad \partial_t u - \Delta u + u\partial_x u = 0,$$

the problem (E) is reduced to the initial value problem for the heat equation via the Hopf-Cole transformation. Indeed, we set

$$\begin{aligned} v(x,t) = &\int_0^t u(y,t)dy + f(t), \\ f'(t) = &\ \partial_x u(0,t) - \frac{u^2(0,t)}{2}, \quad f(0) = 0. \end{aligned}$$

We observe that v satisfies

$$\text{(B)}' \qquad \partial_t v - \Delta v + \frac{1}{2}(\partial_x v)^2 = 0.$$

We set $w(x,t) = e^{\frac{1}{2}v(x,t)}$ and observe that w satisfies the heat equation

$$\partial_t w - \Delta w = 0.$$

(The transformation form v to w is called the Hopf-Cole transformation.) Our problem is reduced to the unique solvability of the heat equation with initial data $w \sim e^{ax^2}$ for large x. The solvability and the existence time estimate is easily proved by the explicit solution formula. The uniqueness part is more subtle but it is widely studied for example in [10]. For the viscous Burgers equation our result easily follows from results for the heat equation [9], [10] without a Lipschitz bound for u_0. However, if $n > 1$ or $\mathbf{G}$ is general, this argument evidently fails to apply.

A classical result of Tychonov [9] states that the Cauchy problem for the heat equation has a unique classical solution in

$$\begin{aligned} &\mathcal{E}(\mathbf{R}^n \times [0,T)) = \\ &\left\{ f \in C(\mathbf{R}^n \times [0,T)) \middle|\ \exists a,\ \exists C > 0 \text{ such that } |f(x,t)|e^{-a|x|^2} \le C \right\}. \end{aligned}$$

for a continuous initial data $u_0(x)$ satisfying growth condition

$$|u_0(x)| \le Ce^{a|x|^2}$$

for some positive constants C, a.

Moreover, D. G. Aronson [1] generalized the result of Tychonov for a parabolic operator with variable coefficients

$$Lu = \partial_t u - \sum_{i,j} \partial_i \{A_{ij}(x,t)\partial_j u + A_i(x,t)u\}$$

with suitable conditions for A_{ij} and A_i for u_0 satisfying

$$\int_{\mathbf{R}^n} |u_0(x)|e^{-a|x|^2} dx < \infty$$

for some positive constant a. He proved that there is a unique solution in

$$\begin{aligned}&\mathcal{E}^2(\mathbf{R}^n \times [0,T)) = \\ &\left\{ f \in L^2_{loc}(\mathbf{R}^n \times [0,T)) \,\middle|\, \int_{\mathbf{R}^n \times (0,T)} e^{-a|x|^2} f^2(x,t)dxdt < \infty \text{ for some } a > 0 \right\}.\end{aligned}$$

for $Lu = 0$ with $u|_{t=0} = u_0$.

K. Ishige [7] proved that solvability of Cauchy problem:

$$\begin{cases} \partial_t(|u|^{\beta-1}u) = \operatorname{div}(|\nabla u|^{p-2}\nabla u), \\ |u|^{\beta-1}u(\cdot,0) = \mu(\cdot), \end{cases}$$

for the initial data μ growing at space infinity. There are some more results for nonlinear equations (see e.g. [7], [3]) but these results do not include (E).

A recent paper [6] of A. Gladkov, M. Guedda and R. Kersner studied the unique solvability of

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \lambda \left| \frac{\partial v}{\partial x} \right|^q \text{ in } \mathbf{R} \times (0,T]$$

with $\lambda > 0$, $q > 1$, when initial data v_0 is not necessary bounded. In fact, they proved that if $u_0(x) \le M_0(\alpha_0 + x^2)^{q/[2(q-1)]-\gamma}$ with some positive constant M_0, α, γ. Then there exists a unique local solution on $\mathbf{R} \times (0,T]$ provided that T satisfies

$$T < \frac{M_0^{-(q-1)}}{\lambda(q-1)} \left(\frac{q-1}{q}\right)^q.$$

If $\gamma > 0$, then the solution can be extended globally in time. When $q = 2$ the equation agrees with (B)$'$. So their result qualitatively implies the local existence for the Burgers equation. However, in general their results do not overlap with ours. Like their result it is possible to prove the global existence when the growth order is less than linear. We shall discuss this topic in a forthcoming paper of the second author.

Uniqueness of solutions without imposing growth conditions was recently studied by G. Barles, S. Biton and O. Ley [2] and K.-S Chou and Y.-C. Kwong [4].

However, the class of quasilinear parabolic equations to which their theory applies excludes our equation (E).

Let us give the idea of the proof. If u_0 is bounded, (E) can be solved by the following iteration:

$$u_{k+1}(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\nabla u_k(s) \bullet \mathbf{G}'(u_k(s))ds. \tag{1}$$

But if u_0 is not bounded, it is difficult to solve (E) by the iteration (1). So we use another iteration:

$$u_{k+1}(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\nabla u_{k+1}(s) \bullet \mathbf{G}'(u_k(s))ds. \tag{2}$$

To use this iteration (2) it is necessary to study the solvability of the linear equation with growing coefficients in the transport term:

$$\partial_t v - \Delta v + \nabla v \bullet \mathbf{p} - vq = 0, \tag{3}$$

for $v \in L^\infty(0,T;L^\infty)$, $\mathbf{p} \in L^\infty(0,T;L_1^\infty)$, $q \in L^\infty(0,T;L^\infty)$. Fortunately, it is not very difficult to solve the linear equation (3) for initial data $v_0 \in BC$, where BC is the set of all bounded continuous functions and BC_m is defined by

$$BC_m = \left\{ f \in C(\mathbf{R}^n) \,\middle|\, \frac{f(x)}{\langle x\rangle^m} \in BC \right\}.$$

Estimating the heat kernel in (2), we get the estimate:

$$\|u_{k+1}(t)\|_{\infty,1} \le C_T\|u_0\|_{\infty,1} + C_T\int_0^t \|\nabla u_{k+1}(s)\|_\infty \|\mathbf{G}'(u_k(s))\|_{\infty,1}ds. \tag{4}$$

Since u_{n+1} satisfies

$$\partial_t u_{n+1} - \Delta u_{n+1} + \nabla u_{n+1} \bullet \mathbf{G}'(u_n) = 0,$$

$\partial_i u_{n+1}$ satisfies

$$\partial_t(\partial_i u_{n+1}) + \Delta(\partial_i u_{n+1}) + \nabla(\partial_i u_{n+1}) \bullet \mathbf{G}'(u_n) + \nabla u_{n+1} \cdot \mathbf{G}''(u_n)(\partial_i u_n) = 0.$$

The maximum principle for (3) yields

$$\|v\|_\infty \le \|v_0\|_\infty + \int_0^t \|q(s)\|_\infty \|v(s)\|_\infty ds.$$

Applying the above maximum principle for v, we get

$$\|\nabla u_{k+1}(t)\|_\infty \le \|\nabla u_0\|_\infty + C_2\int_0^t \|\nabla u_{k+1}(s)\|_\infty \|\nabla u_k(s)\|_\infty ds.$$

By the Gronwall inequality $\|\nabla u_k(t)\|_\infty$ satisfies

$$\|\nabla u_k(t)\|_\infty \le \frac{\|\nabla u_0\|_\infty}{1 - C_2\|\nabla u_0\|_\infty t} \tag{5}$$

for all k.

By (4) and (5) we see that $\{u_k\}$ is a Cauchy sequence in $L^\infty(0, T_0 - \varepsilon; L^\infty_1)$ for any $\varepsilon \in (0, T)$ so that $u := \lim_{k\to\infty} u_k$ is solution of (E). It is easy to prove the uniqueness of solution of (E) by using the maximum principle for equation (3). The key underlying estimate is an apriori estimate:

$$\|\nabla u(t)\|_\infty \le \|\nabla u_0\|_\infty + C_2 \int_0^t \|\nabla u(s)\|_\infty^2 ds$$

for u of (E) which yields, by the Gronwall inequality (Lemma 3.1), a bound for $\|\nabla u(t)\|_\infty$:

$$\|\nabla u(t)\|_\infty \le \frac{\|\nabla u_0\|_\infty}{1 - C_2\|\nabla u_0\|_\infty t}.$$

It is natural to consider a linearly growing initial data for (E). We conclude this introduction by giving a formal argument to show that at most linearly growing initial data is allowed for existence of a solution. We postulate that $u(x,t) = x^\alpha f(t)$ is a solution of (E). By (E) u must satisfy

$$x^\alpha f'(t) = \alpha(\alpha-1)x^{\alpha-2} f(t) + \alpha x^{\alpha-1} f(t)\mathbf{G}'(x^\alpha f(t)).$$

We observe that the growth of the left hand side is x^α. By the assumption of $\mathbf{G}$ the growth of the right hand side is $x^{2\alpha-1}$. Hence α must satisfy $\alpha \le 2\alpha - 1$ so that $\alpha \le 1$.

2. Estimates for the heat semigroup in weighted space

We recall several elementary properties of the heat kernel

$$G_t(x) = G(x,t) = \frac{1}{\sqrt{4\pi t}^n} e^{-\frac{|x|^2}{4t}}.$$

The next two lemmas are well-known but we give a proof for completeness. For a multi-index $a = (a_1, \cdots, a_n)$ by ∂^a we mean $\partial^a = \partial^{a_1} \cdots \partial^{a_n}$, $\partial_i = \partial/\partial x_i$.

Lemma 2.1 *(Derivatives of heat kernel). Derivatives of G_t are of the form*

$$\partial^a G_t(x) = \left(\prod_{i=1}^n p_{a_i}(x_i, t)\right) G_t(x)$$

with some polynomial of x_i and t^{-1} of the form

$$p_{a_i} = p_{a_i}(x_i, t) = \sum_{m \le n \le a_i} C_{m,n} x_i^m t^{-n}.$$

Proof. It is sufficient to prove in the case of $a = (a_1, 0, \cdots, 0)$.

$$\partial_1 \left(p_{a_1}(x_1,t)G_t(x)\right) = \left(\partial_1 p_{a_1}(x_1,t) - \frac{x_1}{2t}p_{a_1}(x_1,t)\right) G_t(x).$$

A standard induction argument yields Lemma 2.1 (In fact, $p_{a_i}(x_i,t)$ is a constant multiple of $(4t)^{-a_i}(-1)^n H_{a_i}((4t)^{-1/2}x_i)$, where H_{a_i} is the Hermite polynomial defined by $(e^{-s^2})^{(j)} = (-1)^{j+1}H_j(s)e^{-s^2}$) . □

Lemma 2.2 *(Polynomial multiplication). For a multi-index a the identity holds*

$$x^a G_t(x) = \left(\prod_{i=1}^n \sum_{0\le j\le a_i} q_{j,i}(t)\partial_i^j\right) G_t(x)$$

with some polynomial $q_{j,i}(t)$ of the form

$$q_{j,i}(t) = \sum_{j\le k\le a_i} C_k t^k.$$

Proof. It is sufficient to prove in the case of $a = (a_1, 0, \cdots, 0)$.By definition,

$$\begin{aligned} x_1^{a_1+1}G_t(x) &= x_1 \sum_{j\le a_1} q_{j,1}(t)\partial_1^j G_t(x) \\ &= x_1 \sum_{j\le a_1} q_{j,1}(t)p_j(x_1,t)G_t(x) \text{ by lemma 2.1} \\ &= \sum_{j\le a_1} q_{j,i}(t)(-2t(-\frac{x_1}{2t}p_j(x_1,t) \\ &\quad +\partial_1 p_j(x_1,t)) + 2t\partial_1 p_j(x_1,t))G_t(x) \\ &= \sum_{j\le a_1} q_{j,i}(t)\left(-2tp_{j+1}(x_1,t) + 2t\partial_1 p_j(x_1,t)\right) G_t(x) \\ &= \sum_{j\le a_1} 2tq_{j,i}(t)\left(-p_{j+1}(x_1,t) + \partial_1 p_j(x_1,t)\right) G_t(x) \end{aligned}$$

by the proof of lemma 2.1. This yields Lemma 2.2. □

Lemma 2.3 *(Estimate of heat kernel in weighted space). There is a constant $C = C(n)$ such that*

$$\|e^{t\Delta}f\|_{\infty,1} \le C(1+\sqrt{t})\|f\|_{\infty,1}$$

holds for all $f \in L_1^\infty(\mathbf{R}^n)$, $t > 0$.

Proof. An elementary calculation shows that

$$\frac{\langle y\rangle}{\langle x\rangle} = 1 + \sum_{i=1}^n (y_i - x_i)h_{i,x}(y),$$

with $h_{i,z}(y)$ defined by

$$h_{i,z}(y) = \frac{(y_i + z_i)}{\langle z\rangle(\langle y\rangle + \langle z\rangle)}.$$

Clearly, we have

$$\sup_i \sup_z \|h_{i,z}\|_\infty = 1.$$

We now calculate $\langle x\rangle^{-1} e^{t\Delta} f$ to get

$$\begin{aligned}
& \frac{1}{\langle x\rangle} \int_{\mathbf{R}^n} G_t(x-y) f(y) dy \\
= & \int_{\mathbf{R}^n} \frac{\langle y\rangle}{\langle x\rangle} G_t(x-y) \frac{f(y)}{\langle y\rangle} dy \\
= & \int_{\mathbf{R}^n} \left(1 + \sum_{i=1}^n h_{i,x}(y)(x_i - y_i)\right) G_t(x-y) \frac{f(y)}{\langle y\rangle} dy \\
= & \int_{\mathbf{R}^n} G_t(x-y) \frac{f(y)}{\langle y\rangle} dy + \int_{\mathbf{R}^n} 2t \sum_{i=1}^n h_{i,x}(y) \partial_i G_t(x-y) \frac{f(y)}{\langle y\rangle} dy.
\end{aligned}$$

Estimating L^∞-norm we obtain

$$\begin{aligned}
\|e^{t\Delta} f\|_{\infty,1} & \leq \|f\|_{\infty,1} + 2tn \sup_i \sup_z \|\partial i e^{t\Delta}(h_{i,z} < x >^{-1} f)\|_\infty \\
& \leq \|f\|_{\infty,1} + 2tn \frac{2}{\sqrt{4\pi t}} \|f\|_{\infty,1} \\
& = \|f\|_{\infty,1} + \frac{2n}{\sqrt{\pi}} \sqrt{t} \|f\|_{\infty,1} \\
& \leq C(1 + \sqrt{t}) \|f\|_{\infty,1}.
\end{aligned}$$

Here we have used $L^\infty - L^\infty$ estimates: $\|e^{t\Delta} k\|_\infty \leq \|k\|_\infty$, $\|\partial i e^{t\Delta} k\|_\infty \leq \pi t^{-1/2} \|k\|_\infty$, with some $C > 0$ independent of k. □

A similar argument yields estimates of derivatives in weighted spaces.

Corollary 2.4. *There is a constant $C = C(n, m, a)$ such that*

$$\|\partial^a e^{t\Delta} f\|_{\infty,m} \leq C \sum_{k=0}^m t^{\frac{k}{2} - \frac{|a|}{2}} \|f\|_{\infty,m}$$

holds for all $f \in L^\infty_m$ and $t > 0$.

Lemma 2.5 *(Hölder continuity of the heat kernel in weighted space). There is a constant $C = C(n, \alpha)$ such that*

$$\|e^{t\Delta} f - e^{s\Delta} f\|_{\infty,1} \leq C \left((t-s)^\alpha s^{-\alpha} + (t-s) t^{-1/2}\right) \|f\|_{\infty,1}$$

holds for all $0 < s \le t$ *and* $0 < \alpha \le 1$.

Proof. We set $g_m(x) = \langle x \rangle^{-m}$. In a similar way of proving Lemma 2.3 we have, by $||e^{t\Delta}k - e^{s\Delta}k||_\infty, s^{1/2}||\partial i e^{t\Delta}k - \partial i e^{s\Delta}k||_\infty \le C(t-s)^\alpha s^{-\alpha}||k||_\infty$

$$\begin{aligned}
&\|e^{t\Delta}f - e^{s\Delta}f\|_{\infty,1} \\
\le\ & \| \left(e^{t\Delta} - e^{s\Delta}\right)(g_1 f)\|_\infty + 2n \sup_i \sup_z \| \left(t\nabla e^{t\Delta} - s\nabla e^{s\Delta}\right)(h_{i,z} g_1 f)\|_\infty \\
\le\ & \| \left(e^{t\Delta} - e^{s\Delta}\right)(g_1 f)\|_\infty \\
& \quad + 2n \sup_i \sup_z (\|(t\partial i e^{t\Delta} - s\partial i e^{t\Delta})(h_{i,z} g_1 f)\|_\infty \\
& \quad + \|(s\partial i e^{t\Delta} - s\partial i e^{s\Delta})(h_{i,z} g_1 f)\|_\infty) \\
\le\ & (C_\alpha (t-s)^\alpha s^{-\alpha} + 2n(C_n(t-s)t^{-1/2} + C_\alpha s(t-s)^\alpha s^{-\alpha-1/2}))\|f\|_{\infty,1} \\
\le\ & (C_\alpha (t-s)^\alpha s^{-\alpha} + 2n(C_n(t-s)t^{-1/2} + C_\alpha (t-s)^\alpha s^{-\alpha+1/2}))\|f\|_{\infty,1} \\
\le\ & C((t-s)^\alpha s^{-\alpha} + (t-s)t^{-1/2})\|f\|_{\infty,1}. \ \square
\end{aligned}$$

In a similar way of proving Lemma 2.5, we obtain a more general version.

Corollary 2.6. *There is a constant* $C = C(n, m, a, \alpha)$ *such that*

$$\begin{aligned}
&\|\partial^a e^{t\Delta} f - \partial^a e^{s\Delta} f\|_{\infty,m} \\
\le\ & C\left((t-s)^\alpha s^{-\alpha}\sum_{k=0}^{m} s^{k/2} + (t-s)t^{-1/2}\sum_{k=0}^{m-1} t^{k/2}\right) s^{-|a|/2}\|f\|_{\infty,m}.
\end{aligned}$$

holds for all $0 < s \le t$ *and* $0 < \alpha \le 1$.

Remark 2.7. *In this paper we use these estimates in finite time interval* $(0, T)$ *so we give the following version of the estimates in Corollary 2.4 and Corollary 2.6.*

$$\|\partial^a e^{t\Delta} f\|_{\infty,m} \le C_T t^{-|a|/2}\|f\|_{\infty,m} \qquad (0 < \forall t \le T),$$

$$\|\partial^a (e^{t\Delta} - e^{s\Delta}) f\|_{\infty,m} \le C_T((t-s)^\alpha s^{-\alpha} + (t-s)t^{-1/2})s^{-|a|/2}\|f\|_{\infty,m}$$

$$(0 < \forall s \le \forall t \le T).$$

Here C_T *is a constant independent of* f *and* t, s *but may depend on* T.

3. Gronwall type inequalities

In this section we recall several versions of Gronwall type inequalities.

Lemma 3.1. *Assume that* $f \in L^\infty(0, T)$, $g \in L^1(0, T)$, *satisfies* $f, g \ge 0$ *a.e.* t. *Assume that* h *is a positive nondecreasing function on* $(0, \infty)$. *Assume that* c *is a positive constant. Let* H *be a primitive function of* $1/h$.

If f satisfies

$$f(t) \leq c + \int_0^t g(s)h(f(s))ds \quad \text{for a.e.} t \in (0, T),$$

then

$$H(f(t)) - H(c) \leq \int_0^t g(s)ds \quad \text{for a.e.} t \in (0, T).$$

¿From now on we suppress the word "a.e.".

Proof. We set

$$F(t) := c + \int_0^t g(s)h(f(s))ds.$$

Then

$$\frac{d}{dt}F(t) = g(t)h(f(t)) \leq g(t)h(F(t)).$$

Integrating this differential inequality, we get

$$H(F(t)) - H(F(0)) \leq \int_0^t g(s)ds.$$

Since h is a positive, the function H is a monotone increasing function. Thus we conclude that

$$H(f(t)) - H(c) \leq \int_0^t g(s)ds. \ \square$$

Remark 3.2. (1) If H has the inverse, Lemma 3.1 implies

$$f(t) \leq H^{-1}\left(H(c) + \int_0^t g(s)ds\right).$$

In this paper we apply Lemma 3.1 when $h(r) = r^2$ and $H(r) = r$. If $h(r) = r^2$, Lemma 3.1 implies that f satisfies

$$f(t) \leq \frac{c}{1 - c\int_0^t g(s)ds},$$

when c is positive. Of course, we may send c to zero in this case. If $h(r) = r$, Lemma 3.1 implies that

$$f(t) \leq ce^{\int_0^t g(s)ds}.$$

(2) In Lemma 3.1 we assume $f \geq 0$, $h \geq 0$, $c > 0$. However, if h satisfies $h(r) = r$, it is not necessary to assume that $f \geq 0$ and that c is a positive constant. Moreover we may take c as a function.

We shall state it for convenience. Assume that $k \in L^\infty(0,T)$, and $g \in L^1(0,T)$ and $g \geq 0$. Assume that $f \in L^\infty(0,T)$ satisfies

$$f(t) \leq k(t) + \int_0^t g(s)f(s)ds, \quad t \in (0,T).$$

Then f satisfies

$$f(t) \leq k(t) + \int_0^t g(s)k(s)e^{\int_s^t g(\tau)d\tau}ds, \quad t \in (0,T).$$

This inequality is known as the famous Gronwall inequality and it is included in many standard text books.

We shall give an application of the Gronwall inequality.

Lemma 3.3. *Assume that $h \in L^\infty(0,T)$ and that $F : [0,T) \times [0,\infty) \to \mathbf{R}$ is locally bounded and $r \mapsto F(t,r)$ is a nonnegative nondecreasing function for all $t \in [0,T)$. Assume that $k \in L^\infty(0,T)$ ($k \geq 0$) satisfies*

$$k(t) = h(t) + \int_0^t k(s)F(s,k(s))ds, \quad t \in (0,T).$$

Assume that $f, g \in L^\infty(0,T)$ satisfy $f, g \geq 0$ and that

$$f(t) \leq h(t) + \int_0^t f(s)F(s,g(s))ds, \quad t \in (0,T).$$

If g satisfies

$$g(t) \leq k(t),$$

then f satisfies

$$f(t) \leq k(t).$$

Proof. By assumption

$$\begin{aligned} f(t) - k(t) &\leq h(t) + \int_0^t f(s)F(s,g(s))ds - h(t) - \int_0^t k(s)F(s,k(s))ds \\ &\leq \int_0^t (f(s) - k(s))F(s,k(s))ds. \end{aligned}$$

By Remark 3.2 (2) we have

$$f(t) - k(t) \leq 0.\square$$

4. Maximum principle

We prepare a maximum principle for equations with a growing coefficient in the transport term. Our results are by no means optimal but it is enough for our purpose.

Lemma 4.1. *Assume that*

$$\begin{array}{ll} u_0 & \in C(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n), \\ p_i & \in C(\mathbf{R}^n \times [0,T]) \cap L^\infty_1(\mathbf{R}^n \times (0,T)), \ \ 1 \le i \le n, \\ q & \in C(\mathbf{R}^n \times [0,T]). \end{array}$$

Assume that $u \in L^\infty(\mathbf{R}^n \times (0,T)) \cap C([0,T) \times \mathbf{R}^n)$ *is a classical solution of*

$$\begin{cases} \partial_t u - \Delta u + \displaystyle\sum_{i=1}^n p_i \partial_i u + q = 0 \text{ in } \mathbf{R}^n \times (0,T), \\ u|_{t=0} = u_0. \end{cases}$$

Then u *satisfies*

$$\|u(t)\|_\infty \le \|u_0\|_\infty + \int_0^t \|q(s)\|_\infty ds.$$

Proof. We set

$$v(t) = u(t) - \|u_0\|_\infty - \int_0^t \|q(s)\|_\infty ds.$$

Then v satisfy $v(x,0) \le 0$ and

$$\partial_t u - \Delta u + \sum_{i=1}^n p_i \partial_i u + q + \|q(t)\|_\infty = 0,$$

in the distribution sense. We set

$$w(t) = v(t) e^{-t}.$$

Then w satisfies $w(x,0) \le 0$ and

$$\partial_t w - \Delta w + \sum_{i=1}^n p_i \partial_i w + w + e^{-t}(q + \|q(t)\|_\infty) = 0.$$

We set

$$w^\varepsilon = w - \varepsilon \log\langle x \rangle.$$

Then w^ε satisfies $w^\varepsilon(x,0) \le 0$ and

$$\begin{aligned} &\partial_t w^\varepsilon - \Delta w^\varepsilon + \sum_{i=1}^n p_i \partial_{x_i} w^\varepsilon + w^\varepsilon \\ &\quad + \varepsilon \left(\log\langle x \rangle - \sum_{i=1}^n \partial_{x_i}^2 \log\langle x \rangle + \sum_{i=1}^n p_i \partial_{x_i} \log\langle x \rangle \right) \\ &\quad + e^{-t} \left(q + \|q(t)\|_\infty \right) = 0. \end{aligned}$$

Suppose that

$$\sup_{\mathbf{R}^n\times[0,T]} w = \alpha > 0.$$

Then for sufficiently small $\varepsilon_0 > 0$, w^ε satisfies

$$\sup_{\mathbf{R}^n\times[0,T]} w^\varepsilon > \frac{\alpha}{2},$$

for all $0 < \varepsilon < \varepsilon_0$. Since w^ε is negative at space infinity, w^ε has a maximum point $(x_\varepsilon, t_\varepsilon)$, *i.e.*

$$\sup_{\mathbf{R}^n\times[0,T]} w^\varepsilon = w^\varepsilon(x_\varepsilon, t_\varepsilon) > \frac{\alpha}{2}.$$

We are able to take ε small so that

$$\left\| -\sum_{i=1}^n \partial_{x_i}^2 \log\langle x\rangle + \sum_{i=1}^n p_i \partial_{x_i} \log\langle x\rangle \right\|_{L^\infty(\mathbf{R}^n\times[0,T])} < \frac{\alpha}{4\varepsilon},$$

since the left hand side is finite by the assumption of $\mathbf{p} = (p_1, \dots, p_n)$. Since $(x_\varepsilon, t_\varepsilon)$ is a maximum of w^ε, we observe that

$$\begin{aligned}
&\partial_t w^\varepsilon - \Delta w^\varepsilon + \sum_{i=1}^n p_i \partial_{x_i} w^\varepsilon + w^\varepsilon \\
&\quad + \varepsilon\left(\log\langle x\rangle - \sum_{i=1}^n \partial_{x_i}^2 \log\langle x\rangle + \sum_{i=1}^n p_i \partial_{x_i} \log\langle x\rangle\right) \\
&\quad + e^{-t}\left(q + \|q(t)\|_\infty\right) > 0 \quad \text{in } B_\rho(x_\varepsilon, t_\varepsilon)
\end{aligned}$$

for sufficiently small $\rho > 0$, where $B_\rho(x_\varepsilon, t_\varepsilon)$ is a closed ball of radius ρ centered at $(x_\varepsilon, t_\varepsilon) \in \mathbf{R}^n \times (0, T)$. This contradicts the equation for w^ε so we conclude that $w^\varepsilon \le 0$. Sending ε to zero, we have $v(x,t) \le 0$, *i.e.*

$$u(x,t) \le \|u_0\|_\infty + \int_0^t \|q(s)\|_\infty ds.$$

A symmetric argument yields

$$u(x,t) \ge -\|u_0\|_\infty - \int_0^t \|q(s)\|_\infty ds$$

and the proof is now complete. □

5. Linear problem in a weighted space

We prove that solvability of a linear equation

$$(L)\begin{cases} \partial_t u - \Delta u + \displaystyle\sum_{i=1}^{n} p_i \partial_i u + qu = 0, \\ u|_{t=0} = u_0, \end{cases}$$

with growing coefficients at the space infinity.

Definition. By a mild solution of (L) we mean that $u \in C(\mathbf{R}^n \times [0,T))$ satisfies

$$u(t) = e^{t\Delta}u_0 - \int_0^t \nabla e^{(t-s)\Delta} \bullet \mathbf{p}(s)u(s)ds + \int_0^t e^{(t-s)\Delta}(\mathrm{div}\mathbf{p}(s))u(s)ds$$
$$- \int_0^t e^{(t-s)\Delta} q(s)u(s)ds,$$

where $\mathbf{p} = (p_1, \cdots, p_n)$.

Lemma 5.1 *(Existence and uniqueness for bounded initial data). Assume that*

$$\begin{aligned} p_i &\in L^\infty((0,T) : X_B) \cap C(\mathbf{R}^n \times [0,T]), \\ \partial_i p_i,\ q &\in BC(\mathbf{R}^n \times [0,T]), \\ u_0 &\in BC(\mathbf{R}^n), \end{aligned}$$

where BC is a set of bounded continuous functions. Let $\alpha \in (0,1)$ and $m \geq 0$ and assume that

$$p_i,\ \partial_i p_i,\ q \in C^\alpha((0,T) : L_m^\infty).$$

Then (L) has a unique classical solution $u \in BC(\mathbf{R}^n \times [0,T))$.

Proof. Step. 1 (Construction of a mild solution). We construct a mild solution $u \in BC(\mathbf{R}^n \times [0,T))$ for integral equation.

(a) Approximation. Let $\psi \in C_0^\infty(\mathbf{R}^n)$ be a cut off function of the form satisfying

$$\psi(x) = \begin{cases} 1 & (|x| \leq 1), \\ 0 & (|x| \geq 2), \end{cases}$$

and $|\psi(x)| \leq 1$ for all $x \in \mathbf{R}^n$. For $k \in \mathbf{N}$ we set $\psi_k(x) = \psi(x/k)$. We set $\mathbf{p}_k = (p_{1,k}, \cdots, p_{n,k}) := \psi_k \mathbf{p}$. Then $\{\mathbf{p}_k\} \subset C(\mathbf{R}^n \times [0,T])$ is a locally uniformly convergent sequence. Moreover

$$\sup_{i,k,0\leq t\leq T} \|p_{i,k}(t)\|_{X_B} < \infty.$$

Let $u_k \in BC(\mathbf{R}^n \times [0,T))$ be a classical solution of

$$\begin{cases} \partial_t u_k - \Delta u_k + \displaystyle\sum_{i=1}^{n} p_{i,k} \partial_i u_k + qu_k = 0, \\ u_k|_{t=0} = u_0. \end{cases}$$

The unique existence of a classical solution is well-known [8]. By Remark 3.2.(2), u_k satisfies

$$\|u_k(t)\|_\infty \le \|u_0\|_\infty e^{T\|q\|_\infty}, \quad t \in (0,T).$$

Hence

$$\begin{gathered} \exists\{u_l\} \subset \{u_k\},\ \exists u \in L^\infty(\mathbf{R}^n \times (0,T)), \quad s.t. \\ u_l \longrightarrow u \quad \text{in } L^\infty(\mathbf{R}^n \times (0,T)) \ *\text{-weak sense,} \end{gathered}$$

and

$$g_m u_l \longrightarrow g_m u \quad \text{in } L^\infty(\mathbf{R}^n \times (0,T)) \ *\text{-weak sense} \quad \forall m \ge 0,$$

where g_m is defined by $g_m(x) = \langle x \rangle^{-m}$.

(b) Convergence. By definition of $\{p_{i,k}\}$, it is easy to prove that

$$\sup_{k, 0\le t\le T} \|\mathbf{p}_k\|_{X_B} < \infty,$$

$$\begin{aligned} p_{i,k} &\longrightarrow p_i, \\ \partial_i p_{i,k} &\longrightarrow \partial_i p_i; \end{aligned}$$

the convergence is locally uniform. We shall prove

$$\begin{aligned} \int_0^t e^{(t-s)\Delta}(\mathrm{div}\mathbf{p}_l(s))u_l(s)ds \to \int_0^t e^{(t-s)\Delta}(\mathrm{div}\mathbf{p}(s))u(s)ds, \\ \int_0^t e^{(t-s)\Delta}q(s)u_l(s)ds \to \int_0^t e^{(t-s)\Delta}q(s)u(s)ds, \\ F_k(x,t) := g_1 \int_0^t \nabla e^{(t-s)\Delta} \bullet \mathbf{p}_l(s)u_l(s)ds \to \\ g_1 \int_0^t \nabla e^{(t-s)\Delta} \bullet \mathbf{p}(s)u(s)ds =: F(x,t), \end{aligned}$$

as $l \to \infty$, $*$-weakly in $L^\infty(\mathbf{R}^n \times (0,T))$. The first two convergences are easy to prove, so we only give a proof of the last convergence. We observe that

$$\begin{aligned} & \int_0^T \int_{\mathbf{R}^n} \varphi(x,t)F_l(x,t)dxdt \\ = & \int_0^T \int_{\mathbf{R}^n} \varphi(x,t)(g_1(x) \int_0^t \int_{\mathbf{R}^n} (\nabla_x G(x-y,t-s)) \\ & \bullet \mathbf{p}_l(y,s)u_l(y,s)dyds)dxdt \\ = & \int_0^T \int_s^T \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \varphi(x,t)g_1(x)(\nabla_x G(x-y,t-s)) \\ & \bullet \mathbf{p}_l(y,s)u_l(y,s)dxdtdyds \\ = & -\int_0^T \int_{\mathbf{R}^n} \mathbf{p}_l(y,s) \\ & \bullet \left(\int_s^T \int_{\mathbf{R}^n} (\nabla_y G(x-y,t-s))\varphi(x,t)g_1(x)dxdt \right) u_l(y,s)dyds, \end{aligned}$$

for all $\varphi \in C_0^\infty(\mathbf{R}^n \times (0,T))$. Since $\mathbf{p}_l$ converge to $\mathbf{p}$ locally uniform, we see that

$$\begin{aligned} &\mathbf{p}_l(y,s) \bullet \left(\int_s^T \int_{\mathbf{R}^n} (\nabla_y G(x-y,t-s))\varphi(x,t)g_1(x)dxdt \right) \\ \to\quad &\mathbf{p}(y,s) \bullet \left(\int_s^T \int_{\mathbf{R}^n} (\nabla_y G(x-y,t-s))\varphi(x,t)g_1(x)dxdt \right) \end{aligned}$$

strongly in $L^1(\mathbf{R}^n \times (0,T)))$. We thus conclude that

$$\int_0^T \int_{\mathbf{R}^n} \varphi(x,t)F_l(x,t)dxdt \to \int_0^T \int_{\mathbf{R}^n} \varphi(x,t)F(x,t)dxdt,$$

as $l \to \infty$. This uniform bound for $\{u_l\}$ in (a) implies a bound for $\{F_l\}$. Since $C_0^\infty(\mathbf{R}^n \times (0,T))$ is dense in $L^1(\mathbf{R}^n \times (0,T)))$, we now conclude that $F_l \to F$ $(l \to \infty)$ $*$-weakly in $L^\infty(\mathbf{R}^n \times (0,T))$.

Since u_l solves the approximate equation, by our convergence results we observe that the limit of $u \in L^\infty(\mathbf{R}^n \times (0,T))$ satisfies

$$\begin{aligned} u(t) = e^{t\Delta}u_0 - \int_0^t \nabla e^{(t-s)\Delta} \bullet \mathbf{p}(s)u(s)ds + \int_0^t e^{(t-s)\Delta}(\mathrm{div}\mathbf{p}(s))u(s)ds \\ - \int_0^t e^{(t-s)\Delta}q(s)u(s)ds. \end{aligned}$$

Step. 2 (Regularity and continuity). We shall prove the Hölder regularity and continuity for $t > 0$ and continuity at $t = 0$.

By using Corollary 2.6 and the integral equation we see that u satisfies $u \in C^\alpha((0,T); L_1^\infty)$ with $\alpha < 1/2$. Since the initial data $u_0 \in BC(\mathbf{R}^n)$, it is easy to see that $u_0 \in BUC_1(\mathbf{R}^n)$, here

$$BUC_m = \left\{ f \in C(\mathbf{R}^n) \,\middle|\, \frac{f(x)}{\langle x \rangle^m} \in BUC \right\};$$

here BUC is the space of all bounded uniformly continuous functions. Since u solves the integral equation, and since $e^{t\Delta}u_0 \in C([0,T); BUC_1)$, we conclude that $u \in C([0,T); BUC_1)$. Thus $u \in BC(\mathbf{R}^n \times [0,T))$.

Here we have invoked the Hölder regularity assumptions of p_i, $\partial_i p_i$, q. For further regularity see for instance [5]. If $u \in BC(\mathbf{R}^n \times [0,T))$ is a classical solution of (L), then by the maximum principle (Lemma 4.1) we conclude the uniqueness of a solution. □

Remark 5.2. It seems to be difficult to prove the uniqueness directly by estimating integral equation.

Corollary 5.3 *(Existence and uniqueness for growing initial data). Assume that* $\mathbf{p}$ *and* q *fulfill the assumptions of Lemma 5.1. Assume that*

$$u_0 \in BC_m(\mathbf{R}^n),$$

where $BC_m(\mathbf{R}^n)$ *be of the form*

$$BC_m(\mathbf{R}^n) = \left\{ f \in C(\mathbf{R}^n) \;\middle|\; \frac{f(x)}{\langle x \rangle^m} \in BC(\mathbf{R}^n) \right\}.$$

Then (L) *has a unique classical solution* $u \in L^\infty_m(\mathbf{R}^n \times (0,T)) \cap C(\mathbf{R}^n \times [0,T))$.

Proof. We set

$$\begin{aligned}
\tilde{p_i} &= p_i - 2mg_{-2}x_i, \\
\tilde{q} &= q + m\sum_{i=1}^{n} p_i g_{-2} x_i - mg_{-4}((m-2)|x|^2 + ng_2), \\
\tilde{u_0} &= g_m u_0.
\end{aligned}$$

We apply Lemma 5.1 and observe that there is a unique classical solution $\tilde{u}$ of

$$\begin{cases}
\partial_t \tilde{u} - \Delta \tilde{u} + \sum_{i=1}^{n} \tilde{p_i}\partial_i \tilde{u} + \tilde{q}\tilde{u} = 0, \\
u|_{t=0} = \tilde{u_0}.
\end{cases}$$

We set $u = g_{-m}\tilde{u}$, then u is a classical solution of (L). The uniqueness follows from that of $\tilde{u}$.

6. Proof of Theorem

For $p \in L^\infty((0,T);X_B)$ let $u \in L^\infty((0,T);X_B)$ be the solution of

$$\begin{cases}
\partial_t u - \Delta u + \nabla u \bullet \mathbf{G}'(p) = 0, \\
u|_{t=0} = u_0 \in X_B.
\end{cases}$$

The unique existence of u is guaranteed in Corollary 5.2. We denote the mapping $p \mapsto u$ in $L^\infty((0,T);X_B)$ by S. We define a sequence of functions $\{u_k\}_{k=1}^\infty$ in $L^\infty((0,T);X_B)$ by

$$u_1 = e^{t\Delta}u_0 \;\;,\;\; u_{k+1} = S(u_k) \text{ for } k \geq 1.$$

Then, by definition, $\{u_k\}$ satisfies

$$u_{k+1}(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}(\nabla u_{k+1}(s) \bullet \mathbf{G}'(u_k(s)))ds.$$

By the estimate for the heat semigroup in weighted space (Remark 2.7) we observe that

$$\begin{aligned}
&\|u_{k+1}(t)\|_{\infty,1} \\
\leq\ & C_T\|u_0\|_{\infty,1} + C_T C_1 \int_0^t \|\nabla u_{k+1}(s)\|_\infty \langle \|u_k(s)\|_{\infty,1}\rangle ds \\
\leq\ & C_T\left(\|u_0\|_{\infty,1} + C_1 \int_0^t \|\nabla u_{k+1}(s)\|_\infty ds\right) \\
& + C_T C_1 \int_0^t \|\nabla u_{k+1}(s)\|_\infty \|u_k(s)\|_{\infty,1} ds
\end{aligned} \tag{6}$$

Since $\{\partial_j u_k\}$ satisfies

$$\partial_t(\partial_j u_{k+1}) - \Delta(\partial_j u_{k+1}) + \partial_i(\partial_j u_{k+1}) \bullet \mathbf{G}'(u_k) + \partial_j u_k \nabla u_{k+1} \bullet \mathbf{G}''(u_k) = 0,$$

the maximum principle (Lemma 4.1) for $\partial_j u_k$ implies that

$$\|\partial_j u_{k+1}(t)\|_\infty \leq \|\partial_j u_0\|_\infty + \int_0^t \|\partial_j u_{k+1}(s)\|_\infty \|\nabla u_{k+1} \bullet \mathbf{G}''(u_k)\|_\infty ds.$$

Multiplying both sides with $\xi_j \geq 0$ and taking the summation over j, we obtain

$$\begin{aligned}
&\textstyle\sum_{j=1}^n \|\partial_j u_{k+1}(t)\|_\infty \xi_j \leq \sum_{j=1}^n \|\partial_j u_0\|_\infty \xi_j \\
&+ \int_0^t \sum_{j=1}^n \|\partial_j u_{k+1}(s)\|_\infty \xi_j \|\nabla u_{k+1} \bullet \mathbf{G}''(u_k)\|_\infty ds.
\end{aligned} \tag{7}$$

We take the supremum of $\xi = (\xi_1, \ldots, \xi_n)$, $|\xi| = 1$, where $|\xi| = (\sum_{j=1}^n |\xi_j|^2)^{1/2}$ to get

$$\|\nabla u_{k+1}(t)\|_\infty \leq \|\nabla u_0\|_\infty + C_2 \int_0^t \|\nabla u_{k+1}(s)\|_\infty \|\nabla u_k(s)\|_\infty ds.$$

by the Schwarz inequality. We now apply Lemma 3.3 with $F(t,r) = C_2 r$, $h(t) = \|\nabla u_0\|_\infty$, $f(t) = \|\nabla u_{k+1}(t)\|_\infty$, $g(t) = \|u_k(t)\|_\infty$. By this choice of F we observe that

$$k(t) = \frac{\|\nabla u_0\|_\infty}{1 - C_2\|\nabla u_0\|_\infty t}.$$

Applying Lemma 3.3 with (7) inductivity, we conclude that $\|\nabla u_k(t)\|_\infty \leq k(t)$, $t \in (0,T)$ for all $k \in \mathbf{N}$, provided that $T < 1/C_2\|\nabla u_0\|_\infty$. By (6) we see that

$$\|u_{k+1}\|_{\infty,1} \leq h(t) + C_1 C_T \int_0^t k(s)\|u_k(s)\|_{\infty,1} ds$$

with

$$h(t) = C_T\|u_0\|_{\infty,1} + C_T C_1 \int_0^t k(s) ds$$

We again apply Lemma 3.3 with $F(t,r) = k(t)r$, $f(t) = \|u_{k+1}\|_{\infty,1}$, $g(t) = \|u_k\|_{\infty,1}$ and conclude that

$$\|u_k\|_{\infty,1} \leq h(t) + \int_0^t k(s)h(s)e^{\int_0^s k(\tau)d\tau} ds$$

Thus we conclude that $\|u_k(t)\|_{\infty,1} \leq k(t)$, $t \in (0,T)$ for all $k \in \mathbf{N}$ provided that $T < 1/C_2\|\nabla u_0\|_\infty$. We observe that $\{u_k\}$ is bounded in $L^\infty((0,T);X_B)$ if $T < 1/C_2\|\nabla u_0\|_\infty$.

We shall estimate the deference:

$$w_k = u_{k+1} - u_k.$$

By definition w_k satisfies

$$\partial_t w_k - \Delta w_k + \nabla w_k \bullet \mathbf{G}'(u_k) + \nabla u_k \bullet (\mathbf{G}'(u_k) - \mathbf{G}'(u_{k-1})) = 0.$$

We change the dependent variable by

$$\tilde{w}_k = \frac{w_k}{\langle x\rangle}$$

and observe that

$$\begin{aligned}
&\partial_t \tilde{w}_k - \Delta \tilde{w}_k + \left(\frac{x}{\langle x\rangle} + \mathbf{G}'(u_k)\right) \bullet \nabla \tilde{w}_k \\
&+ \left(\frac{n\langle x\rangle^2 - |x|^2}{\langle x\rangle^4} + \frac{x}{\langle x\rangle} \bullet \frac{\mathbf{G}'(u_k)}{\langle x\rangle}\right) \tilde{w}_k \\
&+ \left(\frac{1}{\langle x\rangle}\mathbf{G}'(\langle x\rangle \tilde{u}_k) - \frac{1}{\langle x\rangle}\mathbf{G}'(\langle x\rangle \tilde{u}_{k-1})\right) \bullet \nabla u_k = 0.
\end{aligned}$$

By the maximum principle (Lemma 4.1) we obtain

$$\|\tilde{w}_k(t)\|_\infty \leq M_1 \int_0^t \|\tilde{w}_{k-1}(s)\|_\infty ds + M_2 \int_0^t \|\tilde{w}_k(s)\|_\infty ds,$$

where M_1 and M_2 are defined by

$$\begin{aligned}
M_1 &= \sup_{k, 0\leq\tau\leq T} \|\nabla u_k(\tau)\|_\infty, \\
M_2 &= n + 1 + nC_1\sqrt{1 + \left(\sup_{k,0\leq\tau\leq T} \|u_k(\tau)\|_{\infty,1}\right)^2} \quad .
\end{aligned}$$

Thus we have

$$\|w_k(t)\|_{\infty,1} \leq M_1 \int_0^t \|w_{k-1}(s)\|_{\infty,1} ds + M_2 \int_0^t \|w_k(s)\|_{\infty,1} ds.$$

By the Gronwall inequality (Remark 3.2(2))

$$\begin{aligned}
\|w_k(t)\|_{\infty,1} &\leq M_1 \int_0^t \|w_{k-1}(s)\|_{\infty,1} ds \\
&+ M_1 M_2 e^{M_2 T} \int_0^t \int_0^s \|w_{k-1}(\tau)\|_{\infty,1} d\tau ds \\
&\leq M_1 \int_0^t |||w_{k-1}|||_{\infty,1,s} ds \\
&+ M_1 M_2 e^{M_2 T} T \int_0^t |||w_{k-1}|||_{\infty,1,s} ds \\
&= \left(M_1 + M_1 M_2 e^{M_2 T} T\right) \int_0^t |||w_{k-1}|||_{\infty,1,s} ds,
\end{aligned}$$

where $|||w_{k-1}|||_{\infty,1,t}$ is defined by

$$|||w_k|||_{\infty,1,t} = \sup_{0\le\tau\le t} \|w_k(\tau)\|_{\infty,1}.$$

Therefore,

$$|||w_k|||_{\infty,1,t} \le \left(M_1 + M_1 M_2 e^{M_2 T} T\right) \int_0^t |||w_{k-1}|||_{\infty,1,s} ds.$$

We thus conclude that $\{u_k\}$ is Cauchy sequence in $L^\infty((0,T);L_1^\infty)$. Let u be its limit.

Since $\{\nabla u_k\}$ is bounded in $L^\infty(\mathbf{R}^n \times (0,T))$, there exists $v \in L^\infty(\mathbf{R}^n \times (0,T))$ and a subsequence $\{\nabla u_l\} \subset \{\nabla u_k\}$, such that v is the limit of $\{\nabla u_l\}$ in $L^\infty(\mathbf{R}^n \times (0,T))$ in $*$-weak sense. Moreover $v = \nabla u$ in distribution sense.

Since u_l converges to u locally uniformly and the ∇u_l converges to ∇u in $*$-weak sense in $L^\infty(\mathbf{R}^n \times (0,T))$. We see that

$$\begin{aligned} &g_1 \int_0^t e^{(t-s)\Delta} \left(\nabla u_l(s) \bullet \mathbf{G}'(u_{l-1}(s))\right) ds \\ &\longrightarrow g_1 \int_0^t e^{(t-s)\Delta} \left(\nabla u(s) \bullet \mathbf{G}'(u(s))\right) ds \end{aligned}$$

as $l \to \infty$, $*$-weakly in $L^\infty(\mathbf{R}^n \times (0,T))$, where $g_1(x) = 1/\langle x \rangle$. The proof of this convergence is similar to that of Lemma 5.1.

We thus conclude that

$$u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} \left(\nabla u(s) \bullet \mathbf{G}'(u(s))\right) ds.$$

In other words u is a mild solution of (E). By Corollary 5.3 we observe that u is a classical solution and $u \in C([0,T);L_1^\infty)$. By the maximum principle (Lemma 4.1) it is easy to prove the uniqueness of a classical solution of (E). (By the way by construction we have $||\nabla u(t)||_\infty \le k(t)$. However, this can be proved directly by estimating the integral equation and applying Remark 3.2(1).) $\square$

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References

1. D. G. Aronson, *Non-negative solutions of linear parabolic equations*, Ann. Scuola. Norm. Sup. Pisa Cl. Sci., **22** (1968), 607-694.

2. G. Barles, S. Biton and O. Ley, *Quasilinear parabolic equations, unbounded solutions and geometrical equation II. Uniqueness without growth conditions and applications to the mean curvature flow in* $\mathbf{R}^2$, preprint.
3. P. Benilan, M. G. Crandall and M. Pierre, *Solution of the porous medium equation under optimal conditions on initial values*, Indiana Univ. Math. J., **33** (1984), 51-87.
4. K.-S. Chou and Y.-C. Kwong, *On quasilinear parabolic equations which admit global solutions for initial data with unrestricted growth*, Calc. Var., **12** (2001), 281-315.
5. S. D. Eidel'man, *Parabolic Systems*, North-Holland, Amsterdam, 1969.
6. A. Gladkov, M. Guedda and R. Kersner, *The Cauchy problem for the KPZ equation with unbounded initial data*, Dolk. Nats. Akad. Nauk Belarusi, **45** (2001), 11-14, 123.
7. K. Ishige, *On the existence of solutions of the Cauchy problem for a doubly nonlinear parabolic equation*, SIAM J. Math. Anal., **27** (1996) 1235-1260.
8. O. A. Ladyzenskaja, V. A. Solonnikov and N. N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, Amer. Math. Soc., Providence, 1968.
9. A. Tychonoff, *Théorèmes d'unicité pour l'équations de la chaleur*, Mat. Sbron., **42** (1935), 199-216.
10. D. V. Widder, *Positive temperature on an infinite rod*, Trans. Amer. Math. Soc., **55** (1944), 85-95.

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